

PRIME KNOTS WHOSE ARC INDEX IS SMALLER THAN THE CROSSING NUMBER

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ABSTRACT. It is known that the arc index of alternating knots is the minimal crossing number plus two and the arc index of prime nonalternating knots is less than or equal to the minimal crossing number. We study some cases when the arc index is strictly less than the minimal crossing number. We also give minimal grid diagrams of some prime nonalternating knots with 13 crossings and 14 crossings whose arc index is the minimal crossing number minus one.

1. ARC PRESENTATION

A link^{*} can be embedded in a book of finitely many half planes in \mathbb{R}^3 so that each half plane intersects the link in a single arc. Such an embedding is called an *arc presentation* of the link. The minimal number of half planes among all arc presentations of a link is called the *arc index* of the link. The arc index of a link L is denoted by $\alpha(L)$.

Suppose we have an arc presentation of a link L . In each half plane containing a single arc of L , we deform the arc into the union of two horizontal arcs and one vertical arc with the two end points fixed. Then we have a new arc presentation of K which looks like the figure in the left of Figure 1. Relaxing the pairs of consecutive horizontal arcs off the axis, we obtain a diagram of L as shown in the right of Figure 1. The new diagram is called a *grid diagram*. A grid diagram is a link diagram which is the union of a finitely many vertical strings and the same number of horizontal strings with the property that at every crossing the vertical string crosses over the horizontal string. The minimal number of vertical strings among all grid diagram of a link is equal to the arc index of the link.

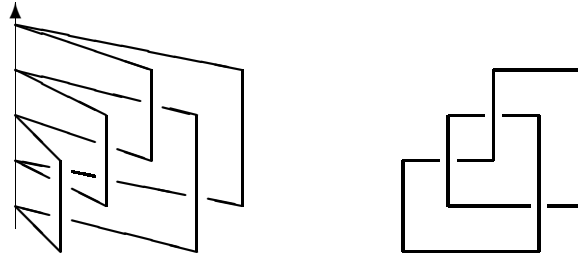


FIGURE 1. An arc presentation of a trefoil knot and its grid diagram

For a link L , let $c(L)$, $F_L(v, z)$, and $\text{spr}_v(F_L(v, z))$ denote the minimal crossing number, the Kauffman polynomial, and the v -spread of $F_L(v, z)$, i.e., the difference between the highest degree and the lowest degree of the variable v in $F_L(v, z)$, respectively. Here we list some of the important known results about the arc index.

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^{*}It is a knot if it has only one component.

Proposition 1.1 (Cromwell). *Every link admits an arc presentation.*

Theorem 1.2 (Cromwell). *If L_1 and L_2 are nontrivial links, then*

$$\alpha(L_1 \# L_2) = \alpha(L_1) + \alpha(L_2) - 2$$

Theorem 1.3 (Bae-Park[†]). *If L is a nonsplit link, then*

$$\alpha(L) \leq c(L) + 2$$

Theorem 1.4 (Morton-Beltrami). *For every link L , we have*

$$\alpha(L) \geq \text{spr}_v(F_L(v, z)) + 2$$

In particular, if L is an alternating link, then

$$\alpha(L) \geq c(L) + 2$$

Theorem 1.5 (Jin-Park[‡]). *A prime link L is nonalternating if and only if*

$$\alpha(L) \leq c(L)$$

Theorem 1.2 allows us to focus on prime links. Theorem 1.3 and Theorem 1.4 together imply that the arc index equals the minimal crossing number plus two for nonsplit alternating links.

Theorem 1.4 and Theorem 1.5 together imply Corollary 1.6 which leads us to conclude that, for prime nonalternating links, if the v -spread of the Kauffman polynomial plus two is equal to the minimal crossing number, then it is equal to the arc index.

Corollary 1.6. *A prime nonalternating link L satisfies the inequality*

$$\text{spr}_v(F_L(v, z)) + 2 \leq \alpha(L) \leq c(L)$$

Table 1 shows the number of prime nonalternating knots up to 16 crossings and those satisfying both equalities in Corollary 1.6.

minimal crossing number n	prime nonalternating knots with n crossings	prime nonalternating knots with n crossings and v -spread $+ 2 = n$
8	3	2
9	8	6
10	42	32
11	185	135
12	888	627
13	5,110	3,250
14	27,436	15,735
15	168,030	83,106
16	1,008,906	423,263

TABLE 1. Nonalternating primes knots whose arc index is determined by Corollary 1.6

In this article, we give three conditions for diagrams of a knot or link to have the arc index smaller than the number of crossings. For each of these conditions we give a list of 13 crossing knots satisfying the condition and having the arc index 12.

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2. THE KNOT-SPOKE DIAGRAM APPROACH DUE TO BAE AND PARK

Now we briefly describe the methods used in the proofs of Theorems 1.3 and 1.5.

A *wheel diagram* is a finite plane graph of straight edges which are incident to a single vertex [1]. The projection of an arc presentation of a knot or a link into the xy -plane is of this shape. For a wheel diagram with n edges to represent a knot or a link, each edge is labeled with an unordered pair of distinct integers, $1, 2, \dots, n$, so that each of the integers appears exactly twice. These numbers indicate the relative z -levels of the end points of the corresponding arcs. Since there are only finitely many choices for labelings, there are only finitely many knots and links for each arc index.



FIGURE 2. Wheel Diagrams

A *knot-spoke diagram* D is a finite connected plane graph satisfying the two conditions:

- (1) There are three kinds of vertices in D ; a *distinguished vertex* v_0 with valency at least four, 4-valent vertices, and 1-valent vertices.
- (2) Every edge incident to a 1-valent vertex is also incident to v_0 . Such an edge is called a *spoke*.

A wheel diagram is a knot-spoke diagram without any non-spoke edges. A knot-spoke diagram D is said to be *prime* if no simple closed curve meeting D in two interior points of edges separates multi-valent vertices into two parts. A multi-valent vertex v of a knot-spoke diagram D is said to be a *cut-point* if there is a simple closed curve S meeting D in the single point v and separating non-spoke edges into two parts.

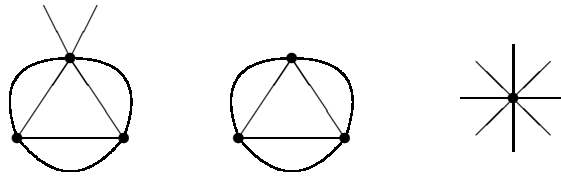


FIGURE 3. Knot-spoke diagrams

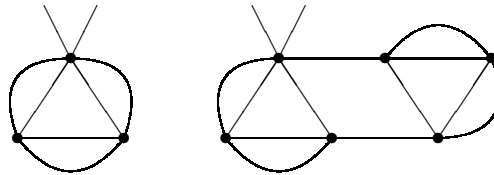


FIGURE 4. Prime diagram and non-prime diagram

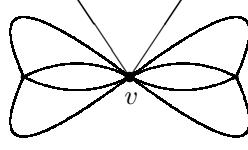
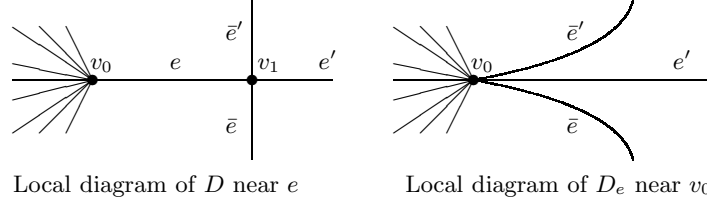


FIGURE 5. A cut-point

Notice that a cut-point free knot-spoke diagram with more than one non-spoke edges cannot have a loop, and that if a prime knot-spoke diagram D has a cut-point, then the distinguished vertex v_0 must be the cut-point with valency bigger than four.

To obtain types of a knot or a link which can be projected onto a knot-spoke diagram D , we need to assign relative heights of the endpoints of edges of D in the following way.

- (3) At every 4-valent vertex, pairs of opposite edges meet in two distinct levels so that a knot-crossing is created.
- (4) If the distinguished vertex v_0 is incident to $2a$ non-spoke edges and b spokes, then its small neighborhood is the projection of $n = a + b$ arcs at distinct levels whose relative z -levels can be specified by the numbers $1, \dots, n$. Every spoke is understood as the projection of an arc on a vertical plane whose endpoints project to v_0 .

FIGURE 6. Contraction of an edge in D

Let e be an edge of a cut-point free knot-spoke diagram D as in Figure 6. The knot-spoke diagram D_e is obtained by contracting e and then replacing each simple loop created from \bar{e} or \bar{e}' by a spoke. The relative z -levels of the edges e' , \bar{e} , \bar{e}' at v_0 in D_e are easily decided by the z -level of e at v_0 and the type of the crossing v_1 so that we do not need to keep track of the z -levels in detail for the proof of Theorem 1.3. But for the proof of Theorem 1.5 we need to pay attention to some spokes corresponding to nonalternating edges.

Lemma 2.1 (Bae-Park). *Let D be a prime knot-spoke diagram without cut-points. Suppose that D has at least two multi-valent vertices. Then there are at least two non-loop non-spoke edges e and f , incident to v_0 , such that the knot-spoke diagrams D_e and D_f have no cut-points.*

A loop in a knot-spoke diagram is said to be *simple* if the other non-spoke edges are in one side of it. By the above lemma, the edge contractions can be performed repeatedly, without creating a cut-point, until we obtain a knot-spoke diagram with $c(D)$ spokes and only one non-spoke edge which is a non-simple loop where $c(D)$ is the number of crossings in D . Notice that the following three properties are preserved.

- (5) D and D_e represent the same knot or link.
- (6) The sum of the number of regions divided by the non-spoke edges and the number of spokes is unchanged.
- (7) D_e is prime if D is prime.

The last non-spoke edge, which is a loop, is being folded to create two extra spokes to show the inequality $\alpha(L) \leq c(L) + 2$ of Theorem 1.3.

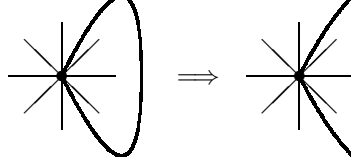


FIGURE 7. Folding the last non-spoke edge

In the case of nonalternating diagrams, there are at least two *removable* spokes so that the inequality of Theorem 1.5 can be proved. The edges to be contracted must be chosen carefully to make nonalternating edges into removable spokes. Therefore a more elaborate method than Lemma 2.1 is needed to avoid cut-point. The following lemma plays an important role for this purpose.

Lemma 2.2 (Jin-Park). *Let D be a prime cut-point free knot-spoke diagram and let e be an edge incident to v_0 and to another 4-valent vertex v_1 such that D_e has a cut-point. Then there exists a simple closed curve S_e satisfying the following conditions.*

- (1) $D_e \cap S_e = v_0$
- (2) S_e separates \bar{e} and \bar{e}' where the four edges incident to v_1 in D are labeled with e, \bar{e}, e', \bar{e}' as in Figure 6.
- (3) S_e separates D_e into two knot-spoke diagrams \bar{D} and \bar{D}' containing \bar{e} and \bar{e}' , respectively. Furthermore \bar{D}' is prime and cut-point free, and there is a sequence of non-spoke edges e_1, e_2, \dots, e_k of D not contained in \bar{D}' such that the knot-spoke diagram $D_{e_1 e_2 \dots e_k}$ is identical with \bar{D}' on non-spoke edges in one side of S_e and has only spokes in the other side.

3. FILTERED SPANNING TREES

Instead of collapsing edges of a diagram D in sequence to obtain a wheel diagram, we consider the tree in D consisting of the edges to be contracted. With this new approach, we describe the method used in the proof of Theorem 1.5.

Let D be a knot diagram. We may consider D as a connected 4-valent plane graph with $c(D)$ vertices and $2c(D)$ edges. A *filtered tree* of D is an increasing sequence $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_k$ of subgraphs of D such that each T_i is a tree containing i edges. The edges of T_k are ordered by the filtering. On the other hand, if the edges of a tree T are ordered so that each of their successive unions is connected, the ordering gives rise to a filtered tree structure on T . If a tree T is prescribed with such an ordering we can consider T as a filtered tree.

The *closure* of T_i , denoted by \bar{T}_i , is the subgraph of D obtained from T_i by adding the edges which are incident to T_i at both ends. An edge f of $\bar{T}_i \setminus T_i \subset D$ is said to be *good* if it meets the edge $e_i = T_i \setminus T_{i-1}$ transversely at the vertex not contained in T_{i-1} . An edge f of $\bar{T}_i \setminus T_i \subset D$ is said to be *bad* if it is an extension of the edge e_i at the vertex not contained in T_{i-1} . In Figure 8, good edges and bad edges are labeled with the letters g and b , respectively.

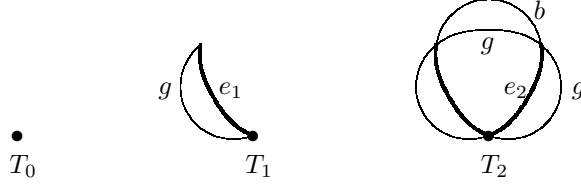


FIGURE 8. A filtered spanning tree and its successive closures

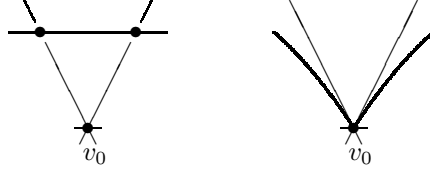


FIGURE 9. Contraction of a triangular region using a doubly-good edge

Let $T_0 \subset T_1 \subset \cdots \subset T_i = T$ be a filtered tree in a diagram D which does not span D . A simple arc Γ , which does not form a bigon together with a single edge of D , is called a *cutting arc* of T if $\Gamma \cap D$ consists of two distinct vertices $p \in T_i \setminus T_{i-1}$ and $c \in T_{i-1}$ such that the simple closed curve in $\Gamma \cup T_i$ separates edges of $D \setminus \overline{T}_i$ into two parts. We say that the filtered tree T is *good* if, for each $j \leq i$, the subtree $T_0 \subset T_1 \subset \cdots \subset T_j$ of T has no cutting arc and \overline{T}_j has no bad edge.

If a filtered tree of D terminates with a spanning tree of D , we call it a *filtered spanning tree* of D . As every spanning tree of D has $c(D) - 1$ edges, a filtered spanning tree of D is of the form $T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_{c(D)-1}$. A filtered spanning tree $T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_{c(D)-1}$ is said to be *good* if each T_i is good for $i < c(D) - 1$ and if $T_{c(D)-1}$ has no cutting arc. Notice that $\overline{T}_{c(D)-1}$ has a bad edge.

We rephrase the statement of Theorem 1.3 in the following way.

Theorem 3.1 (Theorem 1.3 rephrased). *A prime link diagram D admits a good filtered spanning tree and therefore one can obtain an arc presentation with $c(D) + 2$ arcs.*

A good edge $e \in \overline{T}_i \setminus \overline{T}_{i-1}$ is said to be *doubly good* if it is a nonalternating edge and the simple closed curve in $T_i \cup e$ has only good edges of T_j , $j \leq i$, in one

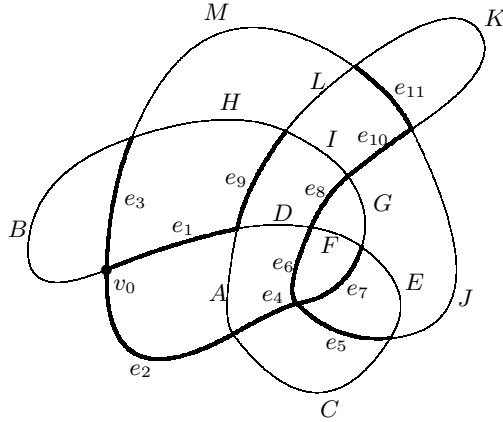


FIGURE 10. Good edges and a bad edge

side. A doubly good edge $e \in \overline{T}_i \setminus \overline{T}_{i-1}$ and the two edges $e_i = T_i \setminus T_{i-1}$, and $e_{i-1} = T_{i-1} \setminus T_{i-2}$ together bound a nonalternating triangular region in D/T_{i-2} , as shown in Figure 9, which can be contracted to reduce the number of regions by one without increasing the number of spokes. Thus the existence of one doubly good edge leads to an arc presentation with one less arcs than the process described in the property (6) of page 5.

Theorem 3.2 (Theorem 1.5 rephrased). *A prime nonalternating diagram D of a link has a good filtered spanning tree which has at least two doubly good edges so that D has an arc presentation with $c(D)$ arcs.*

In Figure 10, the sequence e_1, e_2, \dots, e_{11} gives rise to a filtered spanning tree $T_i = v_0 \cup e_1 \cup \dots \cup e_i$, $i = 0, \dots, 11$. The edges A through L are good and the edge M is bad. The three edges A , F and I are doubly good if they are nonalternating.

The following proposition is immediate from the definition of good filtered trees.

Proposition 3.3. *Let $T_0 \subset T_1 \subset \dots \subset T_m$ be a non-spanning good filtered tree in a prime diagram D . Let e be an edge in D such that $T_m \cap e$ is a single vertex, so that $T_m \cup e$ is a tree. If $T_0 \subset T_1 \subset \dots \subset T_m \subset (T_m \cup e)$ is not a good filtered tree, then $\overline{T_m \cup e}$ has a bad edge or a sufficiently small neighborhood of $\overline{T_m \cup e}$ has disconnected exterior in D .*

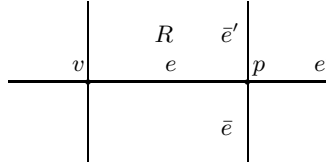


FIGURE 11. Extending T_m along e in D

Suppose that $T_0 \subset T_1 \subset \dots \subset T_m$ and e are as in the hypothesis of Proposition 3.3 and that $T_0 \subset T_1 \subset \dots \subset T_m \subset (T_m \cup e)$ is not a good filtered tree. In Figure 11, v is the vertex of e belonging to T_m and p is the other vertex of e . The three edges e' , \bar{e} , \bar{e}' are incident to e at p and R is a region of D whose boundary contains e and \bar{e}' . Proposition 3.3 implies that there are three cases to consider:

- (B1) e' is a bad edge of $\overline{T_m \cup e}$ joining p and a vertex c of T_m .
- (B2) There is a simple arc Γ_p contained in a single region of D joining p and a vertex c of T_m such that the unique cycle $\overline{\Gamma_p}$ in $T_m \cup e \cup \Gamma_p$ does not enclose the region R but encloses the edges e' and \bar{e} .
- (B3) There is a simple arc Γ_p as in (B2) except that $\overline{\Gamma_p}$ does not enclose e' .

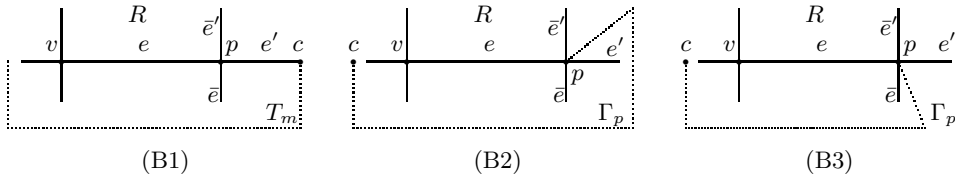


FIGURE 12. Three bad cases for $T_m \cup e$

For each of the above cases, we say that $T_m \cup e$ is a (Bn) -extension of T_m , $n = 1, 2, 3$. If $T_0 \subset T_1 \subset \dots \subset T_m \subset (T_m \cup e)$ is a good filtered (spanning) tree, we say that $T_m \cup e$ is a *good extension* of T_m .

4. MAIN THEOREMS

Before we state our main theorems, we give several lemmas and corollaries. The first two lemmas are translations of the two lemmas written in the language of knot-spoke diagrams into the ones written in the language of filtered trees.

Lemma 4.1 (Lemma 2.1 rephrased). *Let D be a prime knot diagram with $c(D)$ crossings and let $T_0 \subset T_1 \subset \cdots \subset T_m$ be a non-spanning good filtered tree in D . Then there are two edges e and f in $D \setminus \overline{T_m}$ such that $T_m \cup e$ and $T_m \cup f$ are good extensions of T_m .*

Lemma 4.2 (Lemma 2.2 rephrased). *Let $T_0 \subset T_1 \subset \cdots \subset T_m$ be a good filtered tree in a prime diagram D with $m < c(D) - 2$. Let e be an edge in D such that $T_m \cap e$ is a single vertex, say v . Suppose that $T_m \cup e$ is not a good extension of T_m . Then there exists a simple closed curve S satisfying the following conditions.*

- (1) $D \cap S$ is a simple arc which is the union of e and some edges of T_m .
- (2) S separates \bar{e} and \bar{e}' where the four edges incident to p , the endpoint of e other than v , are labeled with e, \bar{e}, e', \bar{e}' as in Figure 11.
- (3) S separates D into two subgraphs \bar{D} and \bar{D}' containing \bar{e} and \bar{e}' , respectively and satisfying $\bar{D} \cap \bar{D}' = D \cap S$. Furthermore there is a sequence e_1, e_2, \dots, e_k of $D \setminus (\bar{D}' \cup \overline{T_m})$ such that $T_0 \subset \cdots \subset T_m \subset T_m \cup e_1 \subset \cdots \subset T_m \cup e_1 \cup \cdots \cup e_k$ is a good filtered tree and $D \setminus \overline{T_m \cup e_1 \cup \cdots \cup e_k} = \bar{D}' \setminus \overline{T_m \cup e}$.

Remark. If $T_m \cup e$ is a (B3)-extension of T_m , the sequence e_1, e_2, \dots, e_k of Lemma 4.2 can be chosen so that $e_k = e$. See the original proof of Lemma 2.2 [6, Proposition 8].

The closure of a region divided by a diagram D is called a *face* of D . Let $T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_j$ be a non-spanning filtered tree of D . If T_j has a cutting arc Γ_p , we may assume that it is *innermost*, in the following sense: Let F be the face of D containing Γ_p and let $\Delta \subset F$ be the disk enclosed by the unique cycle of $T_j \cup \Gamma_p$ satisfying $\partial\Delta \subset \partial F \cup \Gamma_p$. Then any cutting arc of T_j contained in Δ is isotopic to Γ_p .

The following lemma asserts that two innermost cutting arcs are essentially disjoint. We omit the proof.

Lemma 4.3. *Let $T_0 \subset T_1 \subset \cdots \subset T_j$ be a non-spanning filtered tree of D . If Γ_p and Γ_q are innermost cutting arcs of T_i and T_j , respectively, for some $i < j$, then we can isotope Γ_p and Γ_q so that they do not intersect in their interiors.*

If p and q are the two vertices of an edge e of D , we write $e = \overline{pq}$, even in the case that there is another edge joining p and q if we understand which is e .

Lemma 4.4. *Let $T_0 \subset T_1 \subset \cdots \subset T_m$ be a non-spanning good filtered tree of D and let \overline{sp} , \overline{pq} , and \overline{qr} be three consecutive edges along the boundary ∂F of a face F of D . Suppose that $\partial F \cap T_m \neq \emptyset$ and $\overline{pq} \cap T_m = \emptyset$. Then we can always construct a sequence of successive good extensions $T_{m+1} \subset \cdots \subset T_{m+k}$ of T_m such that the closure $\overline{T_{m+k}}$ contains all edges of ∂F except \overline{sp} , \overline{pq} , and \overline{qr} .*

Proof. We construct a sequence of successively extended filtered tree $T_{m+1} \subset \cdots \subset T_{m+k}$ of T_m along the edges of ∂F so that T_{m+k} contains all vertices of ∂F except p and q . If T_{m+i} is not a good extension for some i , we apply Lemma 4.2 to obtain a larger good filtered tree T' not containing the edges \overline{sp} , \overline{pq} , and \overline{qr} , and continue. \square

The following lemma is an immediate consequence of related definitions.

Lemma 4.5. Let $T_0 \subset T_1 \subset \dots \subset T_m$ be a non-spanning good filtered tree of D and let \overline{sp} , \overline{pq} , and \overline{qr} be three consecutive edges along the boundary ∂F of a face F of D . Suppose that \overline{pq} is a nonalternating edge and $\partial F \setminus \{\overline{sp}, \overline{pq}, \overline{qr}\} \subset \overline{T}_m$. Then \overline{pq} becomes a doubly good edge of \overline{T}_{m+2} if the following two conditions are satisfied:

- (1) $T_{m+1} = T_m \cup \overline{sp}$ is a good extension of T_m .
- (2) $T_{m+2} = T_{m+1} \cup \overline{rq}$ is a good extension of T_{m+1} .

Corollary 4.6. Let T_m , F , \overline{sp} , \overline{pq} , and \overline{qr} be as in Lemma 4.5. Suppose that the two conditions below are satisfied:

- (1) $T_{m+1} = T_m \cup \overline{sp}$ is a (B3)-extension of T_m so that there is a tree T' which has a good extension $T' \cup \overline{sp}$ containing T_{m+1} . (See the remark following Lemma 4.2.)
- (2) $T' \cup \overline{sp} \cup \overline{rq}$ is a good extension of $T' \cup \overline{sp}$.

Then \overline{pq} is a doubly good edge of $\overline{T' \cup \overline{sp} \cup \overline{rq}}$.

Lemma 4.7. Let F be a face of a minimal crossing diagram D of a prime knot such that ∂F contains a nonalternating edge \overline{pq} . We may label the vertices of ∂F as $v_0, v_1, \dots, v_{n-2} = q, v_{n-1} = p$, cyclically around F , for some $n \geq 3$. Then there is a good filtered tree $T_0 \subset \dots \subset T_m$ such that $T_0 = v_0$, $v_i \in T_m$, $i = 0, \dots, n-3$ and T_{m+1} is a good extension of T_m along $\overline{v_{n-3}q} \subset \partial F$. Furthermore if the extension T_{m+2} of T_{m+1} along $\overline{v_0p} \subset \partial F$ is not (B3), then \overline{pq} is a doubly good edge of \overline{T}_{m+2} .

Proof. We extend T_0 repeatedly along the edges $\overline{v_{i-1}v_i}$, $i = 1, \dots, n-3$. These extensions are neither (B1) nor (B2) since D is prime. If a (B3)-extension occurs, then, by Lemma 4.2, we can insert more edges before the extension to obtain a good extension along the same edge. Continuing in this manner, we obtain the good extension $T_{m+1} = T_m \cup \overline{v_{n-3}q}$ of T_m . Since D is prime, the extension is neither (B1) nor (B2). This completes the proof. \square

Corollary 4.8. Suppose that the hypothesis of Lemma 4.7 holds and that v_0 and p are two vertices of a bigonal face adjacent to F . Then \overline{pq} is a doubly good edge of \overline{T}_{m+2} .

Proof. In this case, the extension T_{m+2} of T_{m+1} mentioned in Lemma 4.7 cannot be (B3). \square

Let $n \geq 2$. An $(n, 1)$ -tangle is an alternating tangle diagram of $n+1$ crossings whose projection is as shown in Figure 13 (a). A nonalternating knot diagram D is said to be $(n, 1)$ -nonalternating if it can be decomposed of two alternating tangles one of which is an $(n, 1)$ -tangle. Let $n \geq 1$. We can define an n -tangle and n -nonalternating diagram in a similar manner, using Figure 13 (b). A 1-tangle is a single crossing and a 1-nonalternating diagram is also called an *almost alternating diagram*.

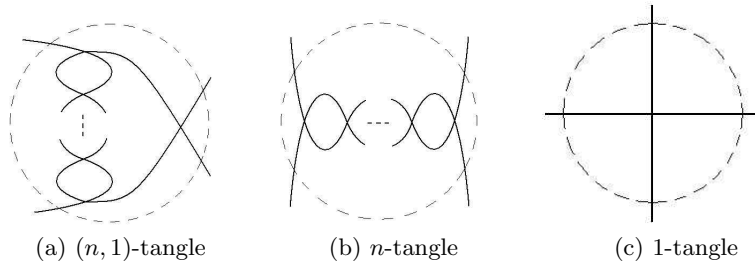


FIGURE 13. Projections of various alternating tangles

Now we are ready to state our main theorems.

Theorem 4.9. *Let $n \geq 2$ and let D be a prime $(n, 1)$ -nonalternating minimal crossing knot diagram having a nonalternating triangular face F_3 . Suppose that faces F_1, F_2, F_3, F , edges e_1, e_2 and a vertex q of D are labeled as in Figure 14. Then $\alpha(D) < c(D)$ if D satisfies the two conditions below:*

- (1) *The face F satisfies $e_1 \cup e_2 \subset \partial F$ and $F \cap (F_1 \cup F_2) = \emptyset$.*
- (2) *There are two vertices $v \in \partial F \cup \{q\}$, $w \in \partial F_2$ and a string a_{vw} of D joining v and w such that no edge of $\partial F_1 \cup \partial F_3$ is contained in a_{vw} .*

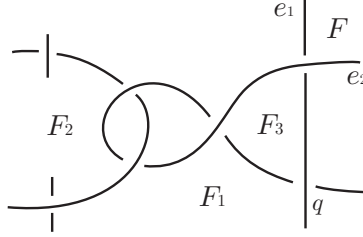


FIGURE 14. $(2, 1)$ -nonalternating diagram

Theorem 4.10. *Let $n \geq 2$ and let D be a prime, n -nonalternating and minimal crossing knot diagram having a nonalternating triangular face F_3 . Suppose that faces F_1, F_2, F_3, F , edges e_1, e_2 and a vertex q of D are labeled as in Figure 15. Then $\alpha(D) < c(D)$ if D satisfies the three conditions below:*

- (1) *The face F satisfies $e_1 \cup e_2 \subset \partial F$ and $F \cap (F_1 \cup F_2) = \emptyset$.*
- (2) *There are two vertices $v \in \partial F$, $w \in \partial F_2 \setminus \{q\}$ and a string a_{vw} of D joining v and w such that no edge of $\partial F_1 \cup \partial F_2 \cup \partial F_3$ is contained in a_{vw} .*
- (3) *∂F_2 consists of at least $n + 3$ edges.*

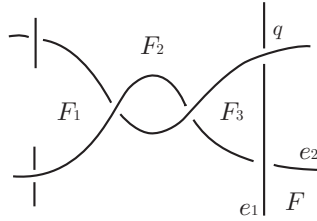


FIGURE 15. 2-nonalternating diagram

Theorem 4.11. *Let D be a prime, almost alternating, and minimal crossing knot diagram having a nonalternating triangular face F_3 . Suppose that faces F_1, F_2, F_3 , an edge e and a vertex q of D are labeled as in Figure 16. Let F be the union of two faces containing e in the intersection of their boundaries. Then $\alpha(D) < c(D)$ if D satisfies the two conditions below:*

- (1) *$F \cap (F_1 \cup F_2) = \{q\}$*
- (2) *There are two vertices $v \in \partial F$, $w \in \partial F_2$ and a string a_{vw} of D joining v and w such that no edge of $\partial F_1 \cup \partial F_3$ is contained in a_{vw} .*

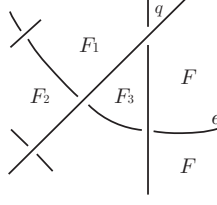


FIGURE 16. almost alternating diagram

5. PROOFS OF MAIN THEOREMS

5.1. Proof of Theorem 4.9. We give a proof of the case $n = 2$. It can be easily adapted for $n > 2$.

Let D' be the diagram obtained from D by a type 3 Reidemeister move over the face F_3 . Some vertices, edges, and faces of D' are labeled as in Figure 17. The vertices v_1, \dots, v_7 are all distinct except in the case that ∂F_1 of D consists of only four edges where $v_1 = v_2$. The two conditions of the theorem are modified to the following conditions on the diagram D' :

- (1') The face F' satisfies $e'_1 \cup e'_2 \subset \partial F'$ and $F' \cap (F'_1 \cup F'_2) = \emptyset$
- (2') There are two vertices $v \in \partial F'$ and $w \in \partial F'_2$ and a string a_{vw} of D joining v and w such that no edge of $\partial F'_1 \cup \partial F'_3$ is contained in a_{vw} . The case $a_{vw} = \overline{v_7 v_5}$ is excluded.

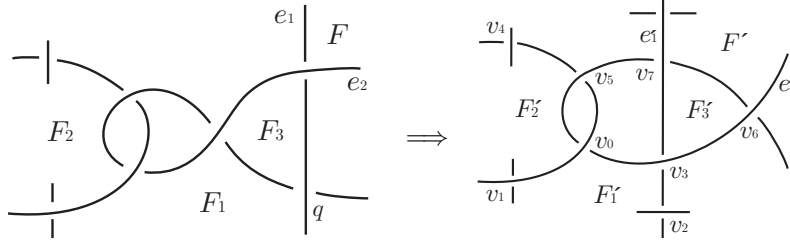


FIGURE 17. A type 3 Reidemeister move

In this proof, we will construct a filtered tree whose successive closures gradually contain $\partial F'_1$, $\partial F'_2$ and $\partial F'_3$ without introducing bad edges and cutting arcs. During the construction, the nonalternating edges $\overline{v_2 v_3}$, $\overline{v_4 v_5}$ and $\overline{v_6 v_7}$ will appear as doubly good edges in this order.

STEP 1. *The edge $\overline{v_2 v_3}$ becomes doubly good.*

Let v'_2 be the vertex of $\partial F'_1$ such that v_0, v_3, v_2, v'_2 are adjacent along $\partial F'_1$ in this order. Applying Lemma 4.7, we obtain a good filtered tree $v_0 = T_0 \subset \dots \subset T_m$ such that T_m contains all vertices of $\partial F'_1$ except v_2 and v_3 , and its good extension T_{m+1} along $\overline{v'_2 v_2}$. Extending once more along $\overline{v_0 v_3}$, we obtain T_{m+2} . The edge $\overline{v_6 v_7}$ obstructs the existence of a (B3) cutting arc for T_{m+2} . By Lemma 4.7, $\overline{v_2 v_3}$ is doubly good in \overline{T}_{m+2} .

STEP 2. *The edge $\overline{v_4 v_5}$ becomes doubly good.*

Let v'_4 be the vertex of $\partial F'_2$ such that v_0, v_5, v_4, v'_4 are adjacent along $\partial F'_2$ in this order. By Lemma 4.4, we have a sequence of good extensions $T_{m+2} \subset \dots \subset T_n$ such that \overline{T}_n contains all edges of $\partial F'_2$ except $\overline{v'_4 v_4}$, $\overline{v_4 v_5}$ and $\overline{v_5 v_6}$. By (2') and by D being

prime and minimal, the extension $T_n \cup \overline{v'_4 v_4}$ cannot be (B1) nor (B2). If it is (B3) then, applying Lemma 4.7, we can replace T_n by a larger tree $T_{n'}$ so that $T_{n'} \cup \overline{v'_4 v_4}$ is a good extension. By the same reasons, the extension $T_{n'+2} = T_{n'} \cup \overline{v'_4 v_4} \cup \overline{v_0 v_5}$ is not (B1) nor (B2). The edge $\overline{v_3 v_7}$ obstructs the existence of a (B3) cutting arc for $T_{n'+2}$. By Lemma 4.7, $\overline{v_4 v_5}$ is doubly good in $\overline{T}_{n'+2}$.

STEP 3. *The edge $\overline{v_6 v_7}$ becomes doubly good.*

By (1') and by D being prime and minimal, the extension $T_{n'+2} \cup \overline{v_3 v_6}$ cannot be (B1) nor (B2). If it is (B3) then, applying Lemma 4.7, we can replace $T_{n'+2}$ by a larger tree $T_{n''}$ so that $T_{n''} \cup \overline{v_3 v_6}$ is a good extension. Now we consider the extension $T_{n''+2} = T_{n''} \cup \overline{v_3 v_6} \cup \overline{v_3 v_7}$. It cannot be (B1) by one of the conditions (1'), (2'), D' being prime and minimal, depending on the location of the endpoint $c \in T_{n''} \cup \overline{v_3 v_6}$ of the bad edge e'_1 . It cannot be (B2) nor (B3) by one of the conditions (1'), (2') and D' being prime, depending on the location of the endpoint $c \in T_{n''} \cup \overline{v_3 v_6}$ of the cutting arc Γ_p where $p = v_7$. By Lemma 4.5 and Corollary 4.6, the nonalternating edge $\overline{v_6 v_7}$ is a doubly good edge of $\overline{T}_{n''+2}$.

5.2. Proof of Theorem 4.10. Let D' be the diagram obtained from D by a type 3 Reidemeister move over the face F_3 . Some vertices and faces of D' are labeled as in Figure 18. The vertices v_0, \dots, v_6 are all distinct. The three conditions of the theorem are modified to the following conditions on the diagram D' :

- (1') The face F' satisfies $e'_1 \cup e'_2 \subset \partial F'$ and $F' \cap (F'_1 \cup F'_2) = \emptyset$
- (2') There are two vertices $v \in \partial F'_1$ and $w \in \partial F'_2$ and a string a_{vw} of D joining v and w such that no edge of $\partial F'_1 \cup \partial F'_2 \cup \partial F'_3$ is contained in a_{vw} . The case $a_{vw} = \overline{v_6 v_2}$ is excluded.
- (3') $\partial F'_2$ consists at least $n + 2$ edges.

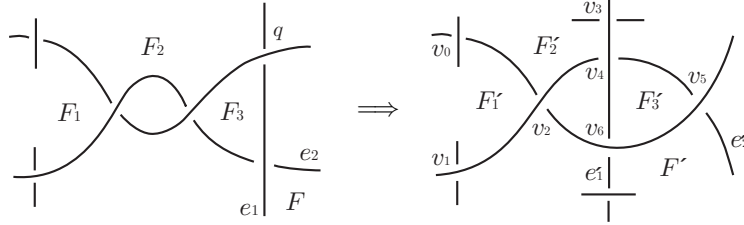


FIGURE 18. A type 3 Reidemeister move

Similarly as in the proof of Theorem 4.9, we construct a filtered tree whose successive closures gradually contain $\partial F'_1, \partial F'_2$ and $\partial F'_3$ without introducing bad edges and cutting arcs. During the construction, the nonalternating edges $\overline{v_1 v_2}$, $\overline{v_3 v_4}$ and $\overline{v_5 v_6}$ will appear as doubly good edges. We skip the detail.

5.3. Proof of Theorem 4.11. Let D' be the diagram obtained from D by a type 3 Reidemeister move over the face F_3 . Some vertices and faces of D' are labeled as in Figure 19. The vertices v_0, \dots, v_5 are all distinct. Let F' be the union of two faces containing e' in the intersection of their boundaries. The two conditions of the theorem are modified to the following conditions on the diagram D' :

- (1') $F' \cap (F'_1 \cup F'_2) = \{v_4\}$
- (2') There are two vertices $v \in \partial F'$ and $w \in \partial F'_2$ and a string a_{vw} of D joining v and w such that no edge of $\partial F'_1 \cup \partial F'_3$ is contained in a_{vw} . The case $a_{vw} = \overline{v_4 v_3}$ is excluded.

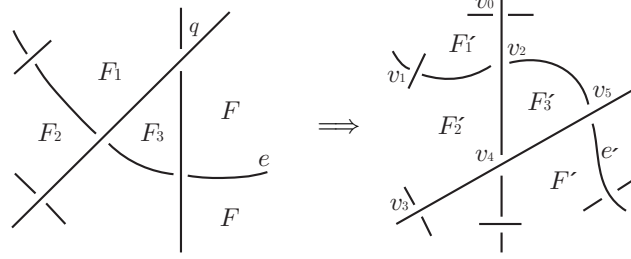
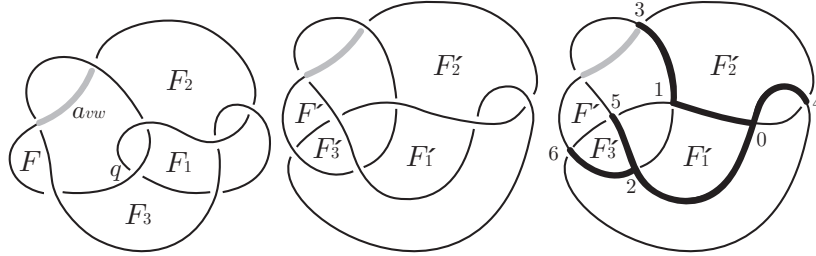
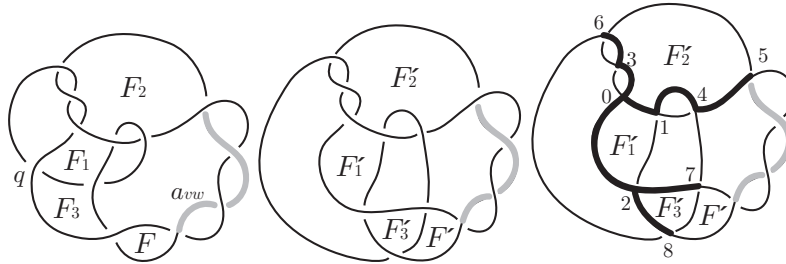


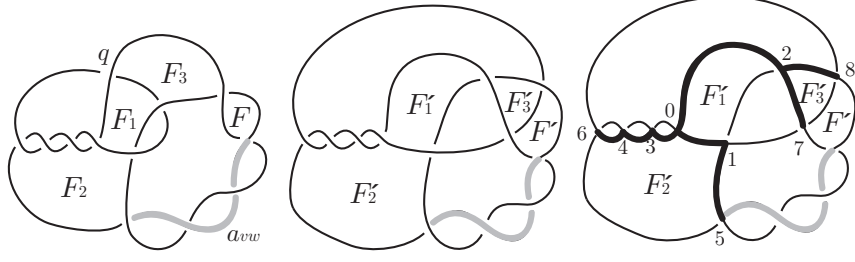
FIGURE 19. A type 3 Reidemeister move

Similarly as in the proof of Theorem 4.9, we construct a filtered tree whose successive closures gradually contain $\partial F'_1$, $\partial F'_2$ and $\partial F'_3$ without introducing bad edges and cutting arcs. During the construction, the nonalternating edges $\overline{v_1 v_2}$, $\overline{v_3 v_4}$ and $\overline{v_4 v_5}$ will appear as doubly good edges. We skip the detail.

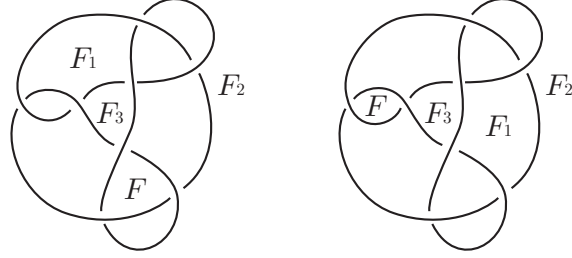
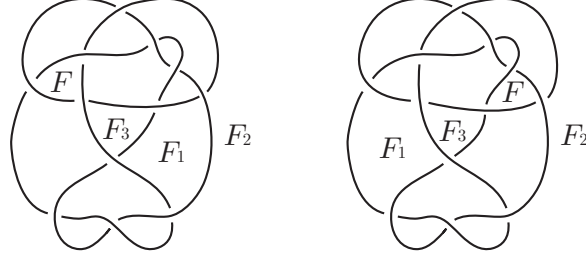
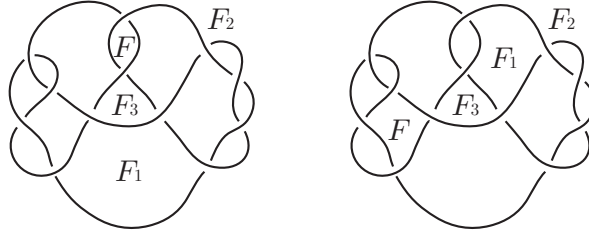
6. EXAMPLES AND NON-EXAMPLES

6.1. Examples of Theorem 4.9. The first of the three diagrams in each figure is the minimal diagram which is $(n, 1)$ -nonalternating. The second is obtained by a type 3 Reidemeister move over the face F_3 . The third is marked with a good filtered tree whose closure has three doubly good edges. If a black-thickened edge is \overline{ij} with $i < j$ then it is the j -th edge of the tree. This comment also applies to the subsection 6.3.


 FIGURE 20. $(2, 1)$ -nonalternating : $\alpha(8n3) = 7$

 FIGURE 21. $(3, 1)$ -nonalternating : $\alpha(12n475) = 10$

FIGURE 22. $(4,1)$ -nonlabeled : $\alpha(12n725) = 10$

6.2. Non-examples of Theorem 4.9. Each figure shows same diagram twice with different choices of faces F_1 , F_2 , F_3 and F . One can check that a condition of the theorem does not hold. This comment also applies to the subsection 6.4.

FIGURE 23. $(2,1)$ -nonlabeled : $\alpha(8n2) = 8$ FIGURE 24. $(3,1)$ -nonlabeled : $\alpha(12n699) = 12$ FIGURE 25. $(4,1)$ -nonlabeled : $\alpha(12n305) = 12$

6.3. Examples of Theorem 4.10.

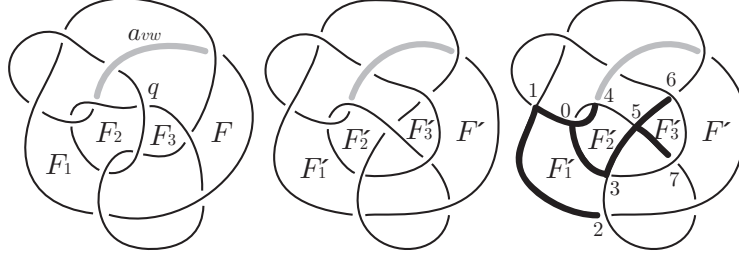


FIGURE 26. 2-nonalternating : $\alpha(12n810) = 11$

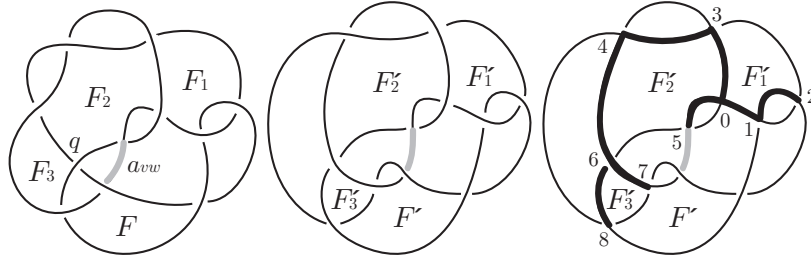


FIGURE 27. 3-nonalternating : $\alpha(11n110) = 10$

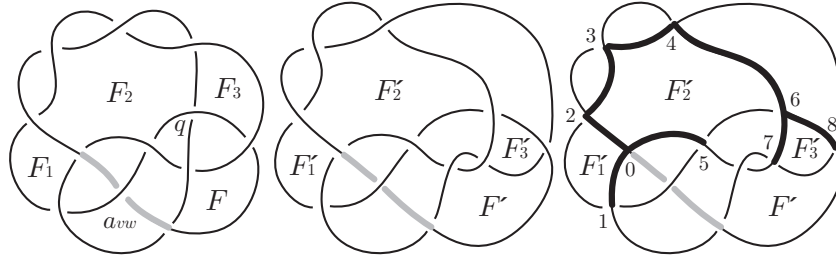


FIGURE 28. 4-nonalternating : $\alpha(12n847) = 11$

6.4. Non-examples of Theorem 4.10.

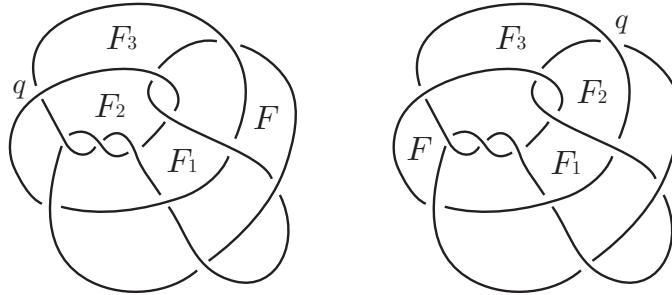
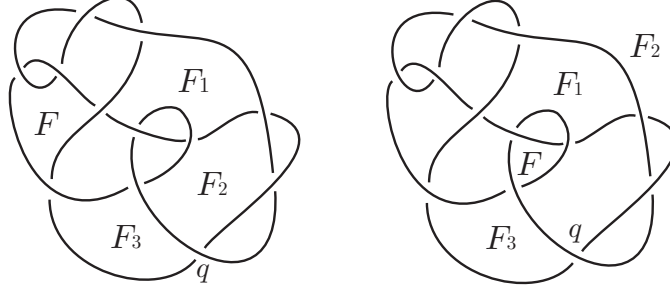
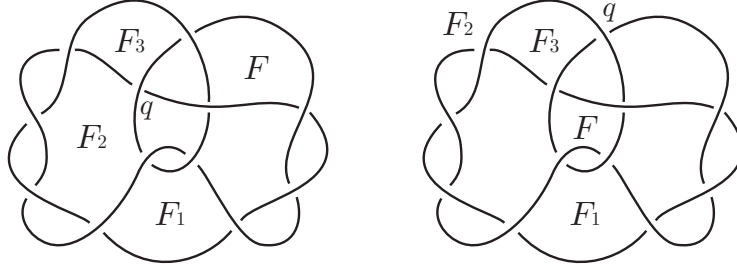


FIGURE 29. 2-nonalternating : $\alpha(12n777) = 12$

FIGURE 30. 3-nonalternating : $\alpha(12n389) = 12$ FIGURE 31. 4-nonalternating : $\alpha(12n767) = 12$

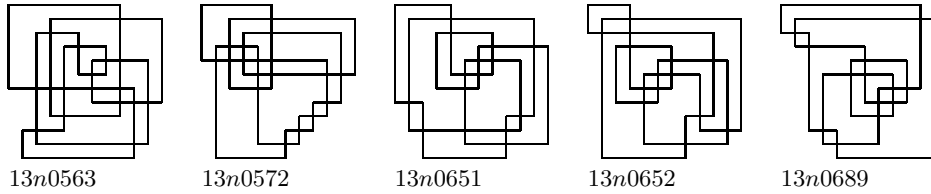
7. NONALTERNATING KNOTS WITH $\alpha(K) = c(K) - 1$

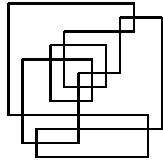
In [11] Nutt identified all knots up to arc index 9. In [2] Beltrami determined arc index for prime knots up to 10 crossings. In [5] Jin et al. identified all prime knots up to arc index 10. In [9] Ng determined arc index for prime knots up to 11 crossings. In [6] Jin and Park identified the prime knots up to arc index 11.

Using the Dowker-Thistlethwaite codes contained in Knotscape [14], we made lists of 13 crossing knots and 14 crossing knots which are $(n, 1)$ -nonalternating, n -nonalternating or almost alternating. Applying the conditions listed in the theorems, we were able to find the lists below.

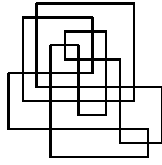
7.1. A partial list of 13 crossing knots with arc index 12. The 13 crossing knots in the lists below do not appear in the article [7] containing all prime knots up to arc index 11. Using the methods described in the proofs of main theorems, we were able to find grid diagrams of them with 12 vertical arcs. In the grid diagrams below, we have the convention that the vertical edges pass over the horizontal edges.

(2, 1)-nonalternating.

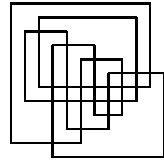




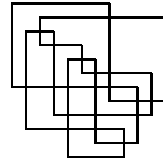
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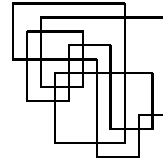
13n0789



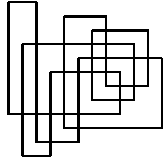
13n0790



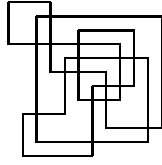
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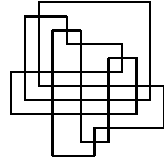
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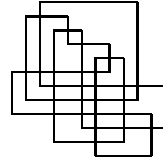
13n1000



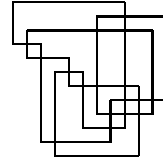
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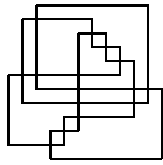
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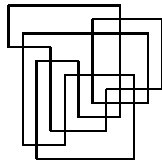
13n1003



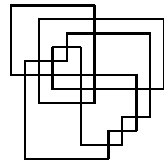
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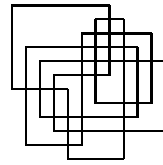
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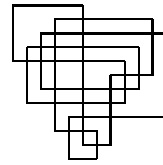
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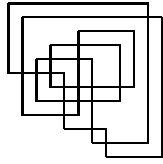
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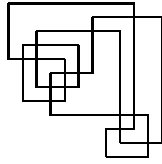
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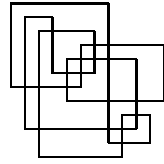
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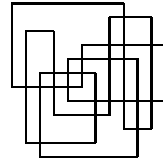
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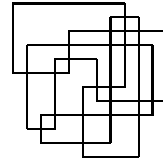
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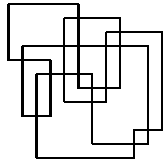
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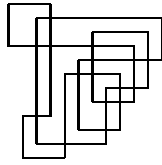
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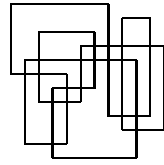
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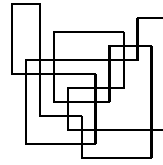
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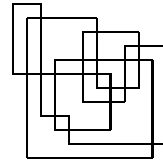
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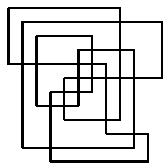
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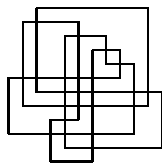
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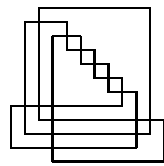
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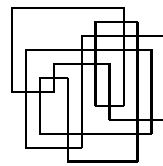
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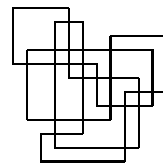
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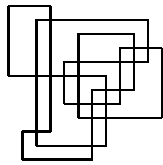
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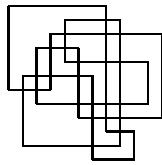
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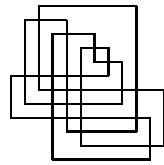
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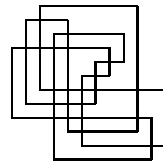
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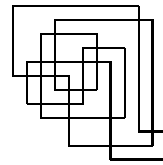
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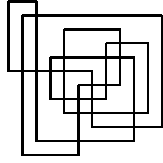
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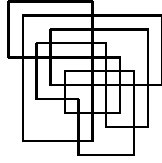
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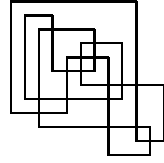
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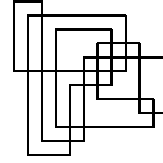
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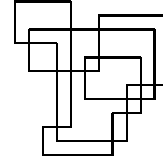
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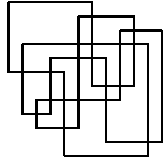
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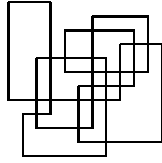
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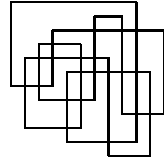
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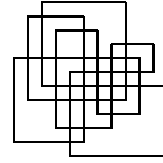
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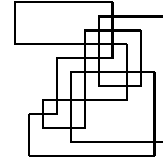
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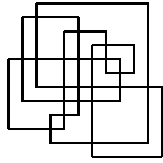
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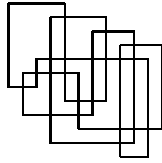
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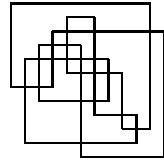
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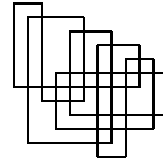
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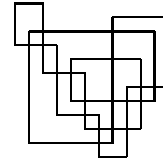
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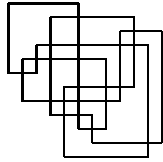
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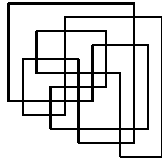
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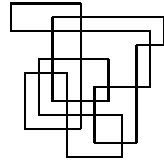
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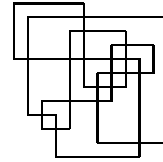
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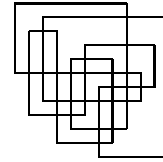
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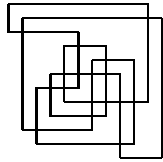
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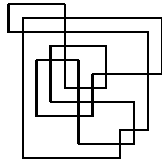
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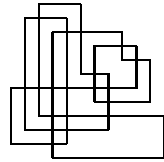
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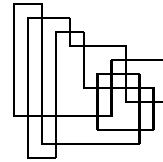
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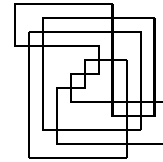
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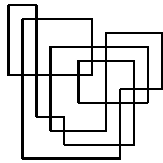
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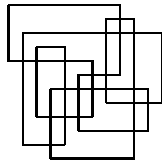
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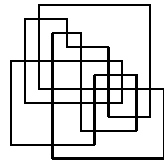
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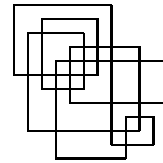
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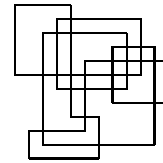
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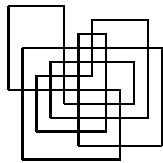
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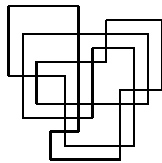
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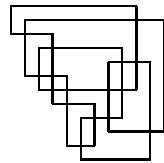
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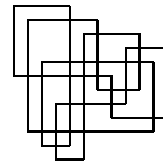
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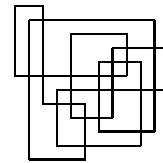
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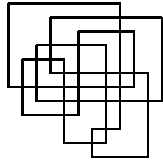
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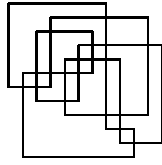
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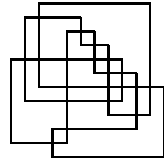
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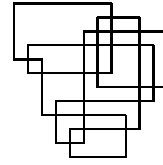
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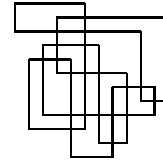
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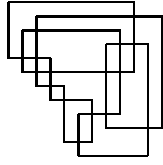
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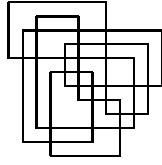
13n2126



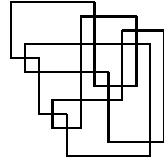
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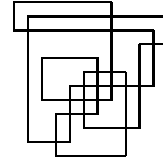
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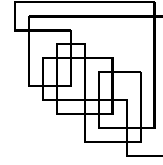
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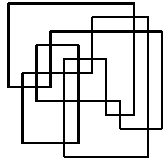
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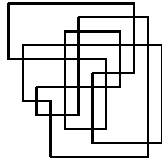
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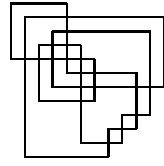
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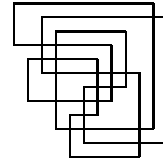
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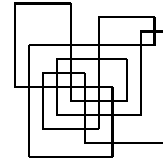
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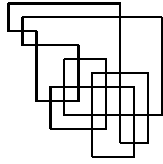
13n2181



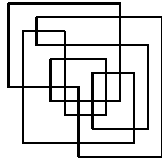
13n2182



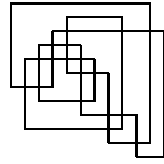
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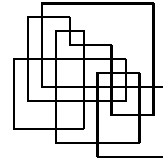
13n2204



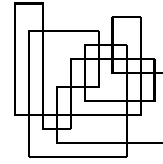
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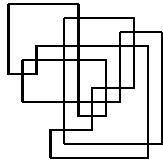
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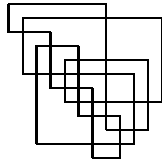
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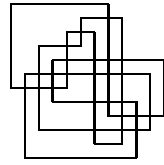
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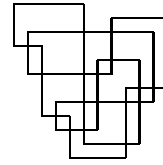
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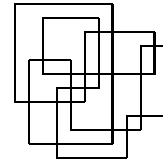
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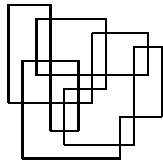
13n2251



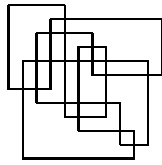
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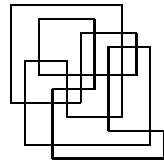
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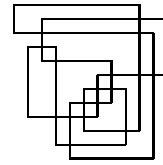
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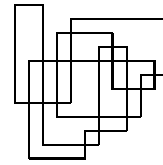
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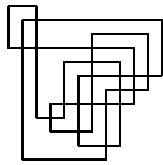
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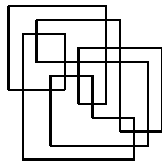
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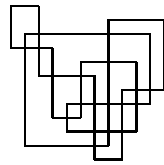
13n2296



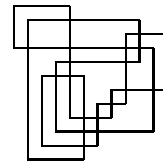
13n2301



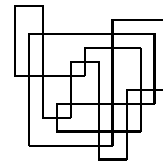
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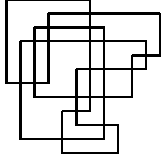
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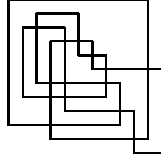
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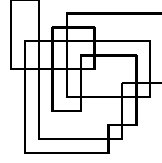
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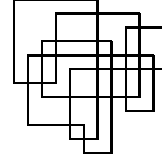
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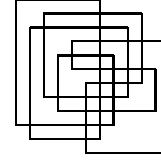
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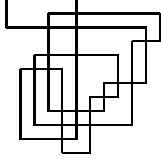
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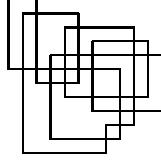
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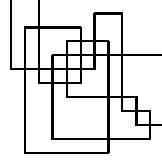
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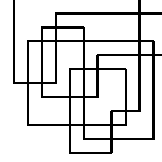
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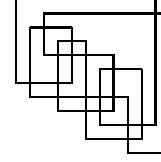
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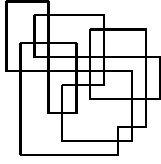
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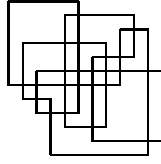
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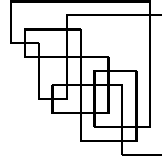
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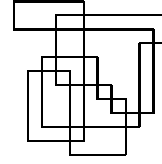
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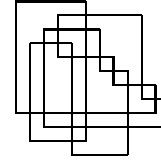
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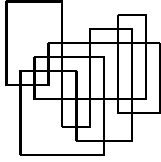
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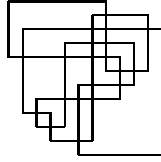
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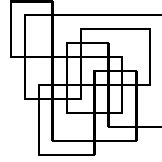
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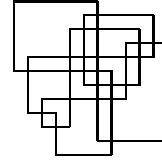
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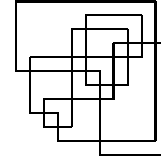
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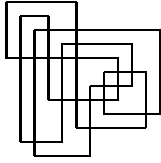
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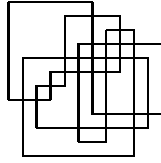
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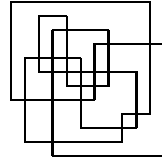
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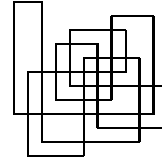
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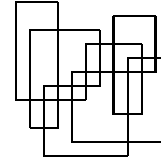
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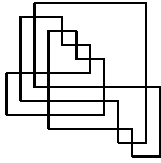
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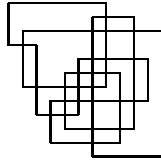
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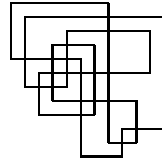
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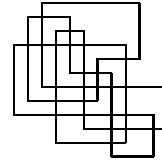
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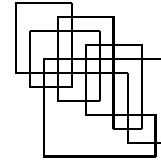
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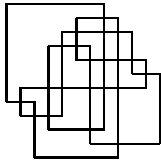
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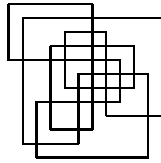
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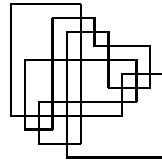
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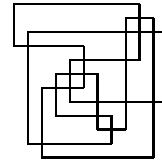
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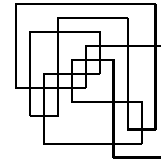
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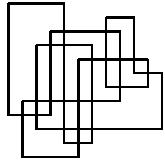
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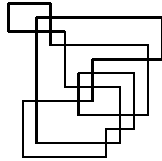
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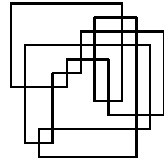
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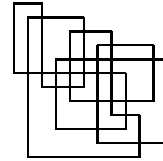
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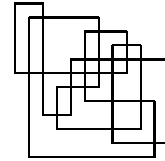
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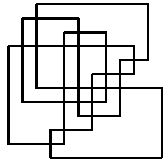
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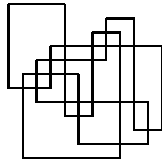
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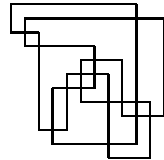
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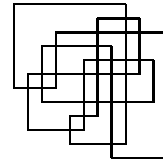
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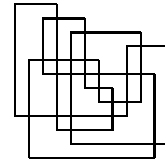
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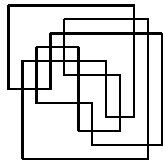
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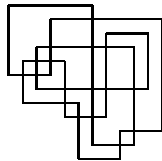
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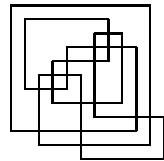
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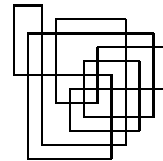
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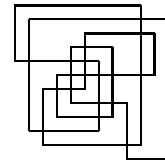
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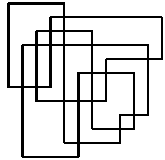
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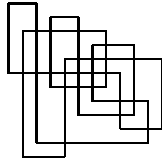
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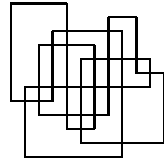
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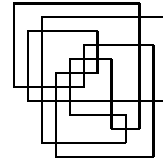
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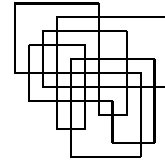
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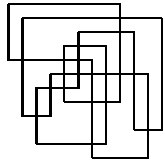
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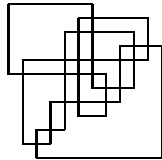
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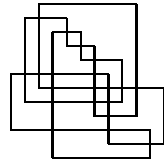
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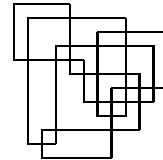
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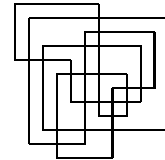
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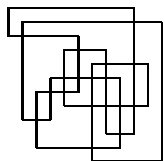
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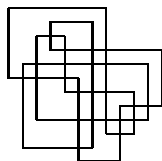
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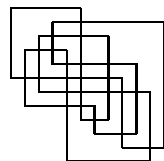
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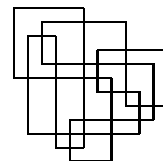
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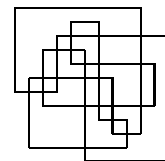
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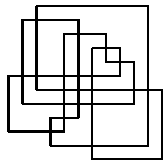
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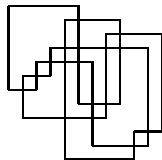
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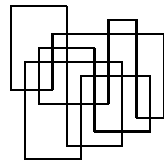
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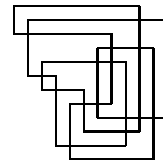
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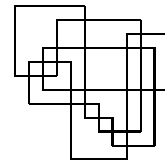
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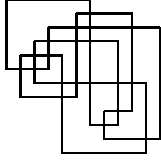
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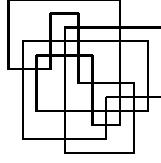
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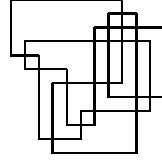
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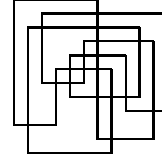
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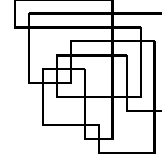
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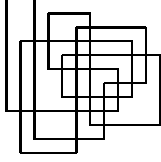
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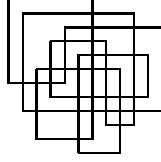
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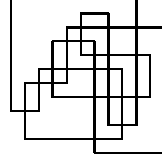
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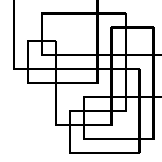
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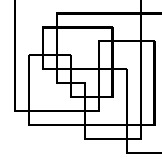
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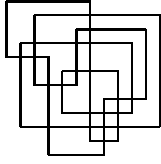
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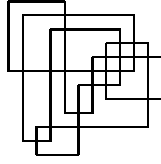
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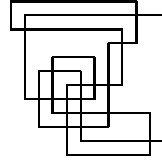
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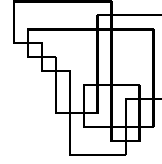
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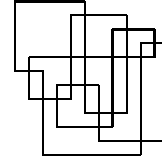
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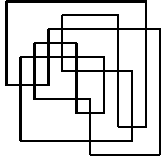
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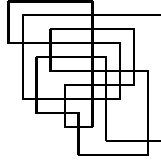
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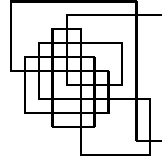
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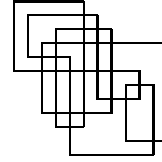
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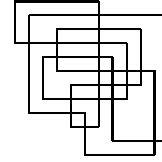
13n2995



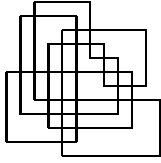
13n2999



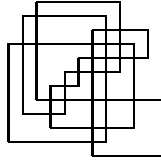
13n3000



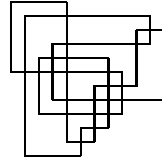
13n3001



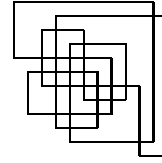
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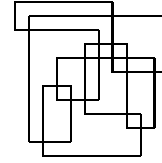
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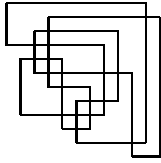
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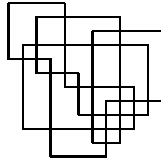
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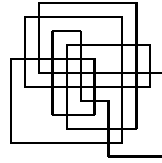
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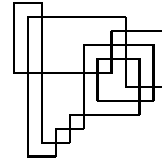
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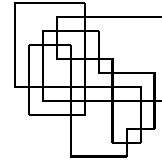
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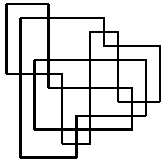
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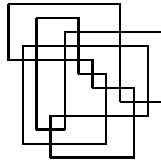
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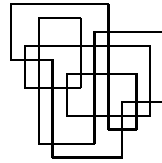
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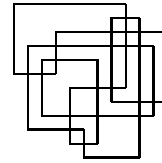
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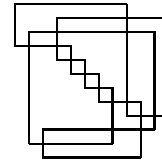
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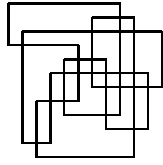
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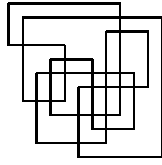
13n3086



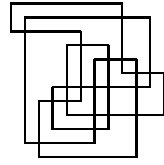
13n3087



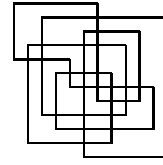
13n3096



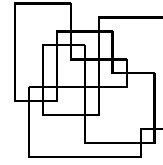
13n3098



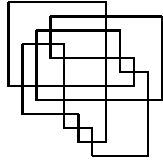
13n3118



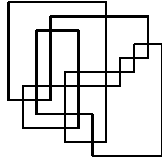
13n3119



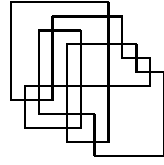
13n3124



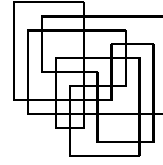
13n3127



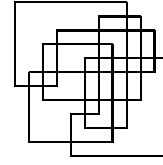
13n3142



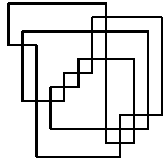
13n3144



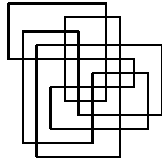
13n3149



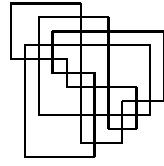
13n3155



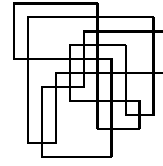
13n3163



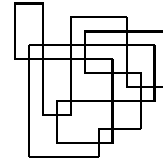
13n3175



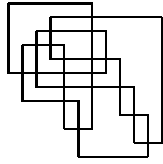
13n3185



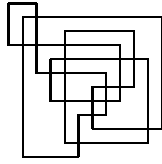
13n3211



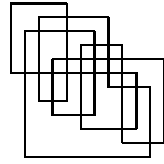
13n3226



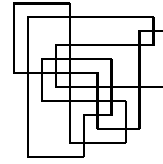
13n3227



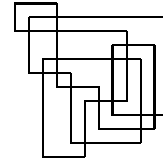
13n3228



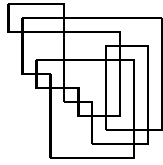
13n3234



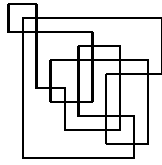
13n3238



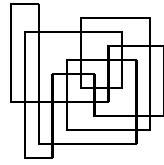
13n3252



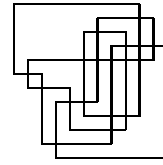
13n3255



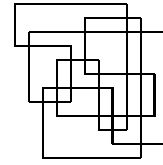
13n3256



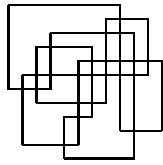
13n3257



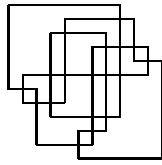
13n3258



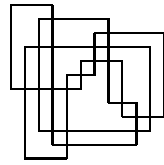
13n3288



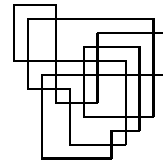
13n3297



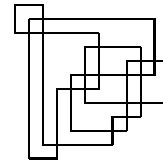
13n3298



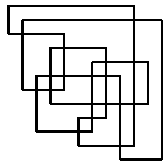
13n3301



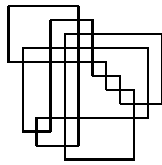
13n3309



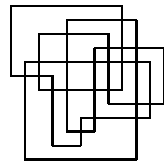
13n3320



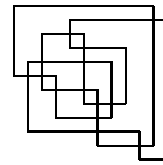
13n3321



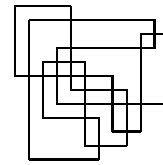
13n3335



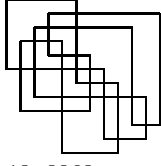
13n3343



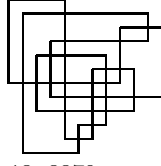
13n3347



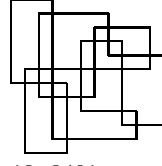
13n3350



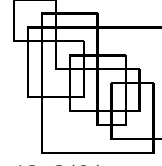
13n3363



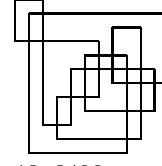
13n3370



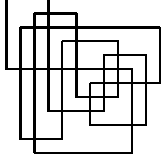
13n3401



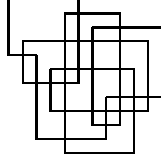
13n3404



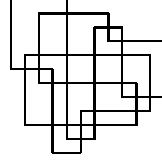
13n3406



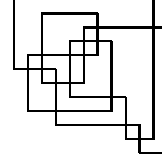
13n3413



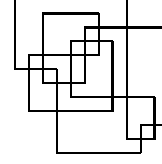
13n3428



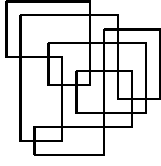
13n3450



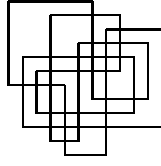
13n3465



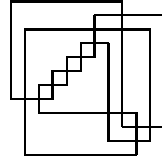
13n3468



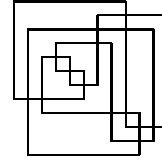
13n3477



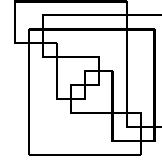
13n3484



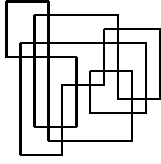
13n3501



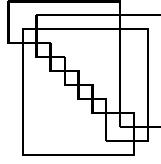
13n3503



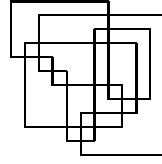
13n3514



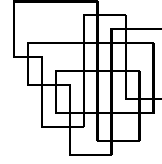
13n3518



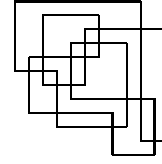
13n3521



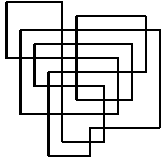
13n3534



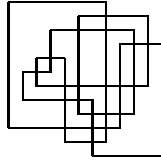
13n3541



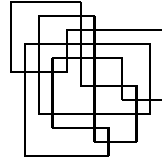
13n3550



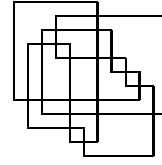
13n3553



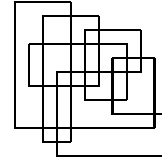
13n3554



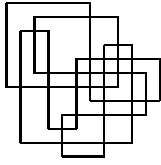
13n3555



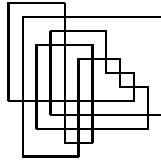
13n3558



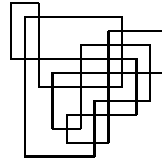
13n3562



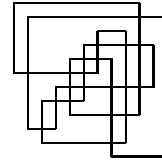
13n3581



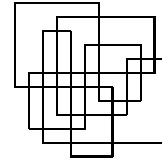
13n3594



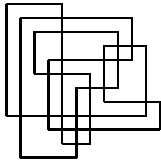
13n3597



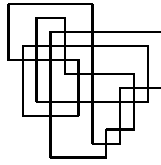
13n3600



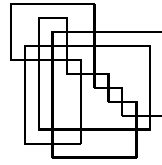
13n3611



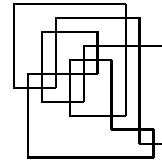
13n3617



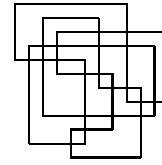
13n3673



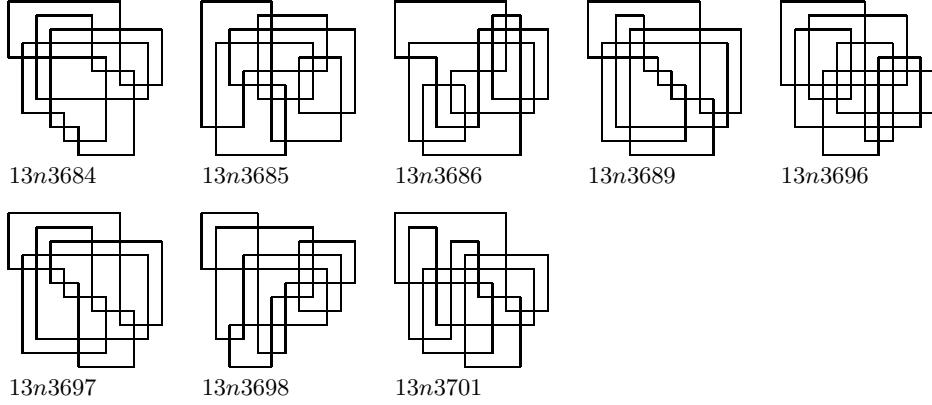
13n3676



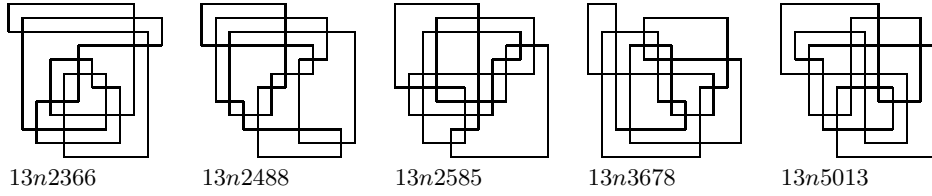
13n3680



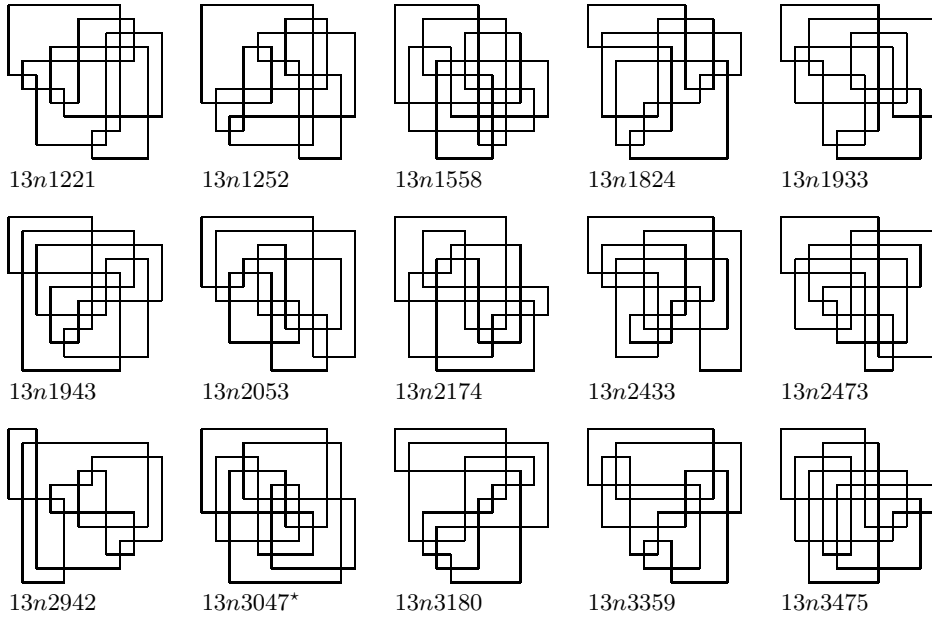
13n3683



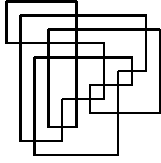
(3, 1)-nonalternating.



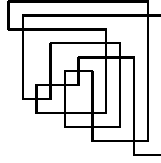
2-nonalternating[§]



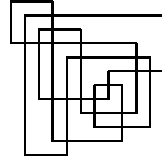
[§]Those marked with * do not satisfy some conditions of Theorem 4.10, but satisfy $\alpha(D) < c(D)$.



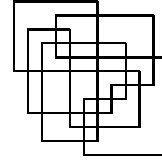
13n3482



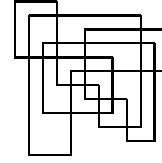
13n3529



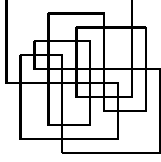
13n3568



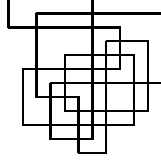
13n3700



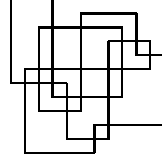
13n4159



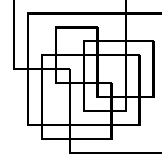
13n4177



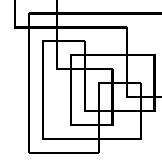
13n4202



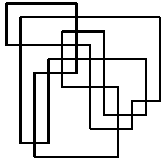
13n4229



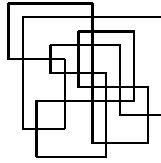
13n4231



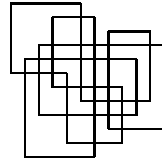
13n4235



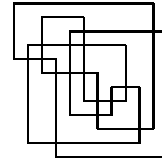
13n4236



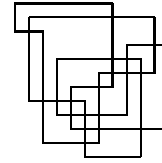
13n4258



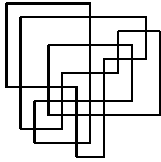
13n4276



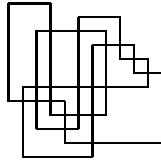
13n4308



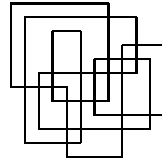
13n4333



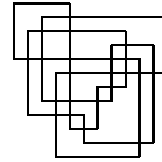
13n4336



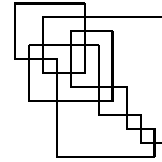
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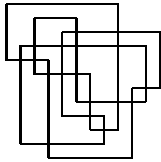
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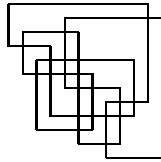
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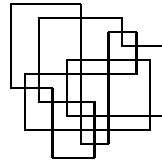
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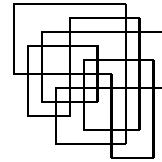
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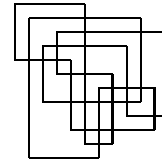
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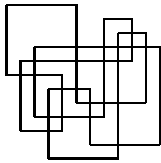
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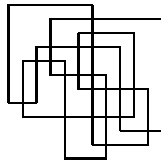
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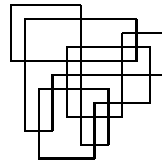
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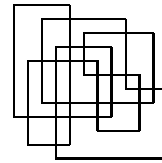
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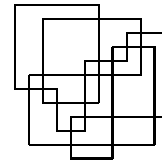
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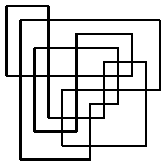
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13n4895

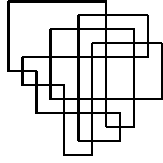


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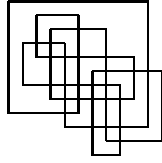


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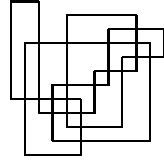
3-nonalternating.



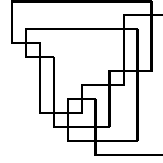
13n806



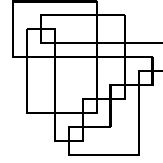
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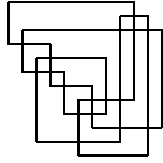
13n2128



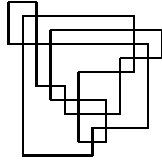
13n2421



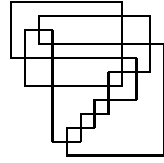
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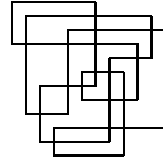
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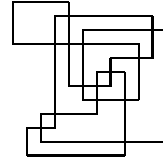
13n2661



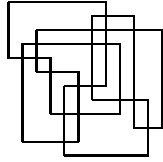
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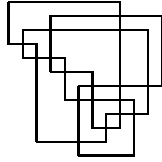
13n3207



13n3807

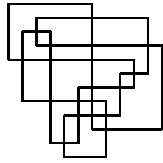


13n4380

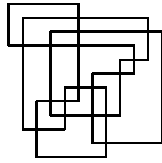


13n4430

4-nonalternating.

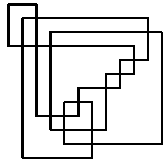


13n3575



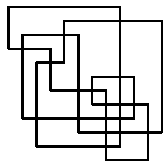
13n5014

5-nonalternating.

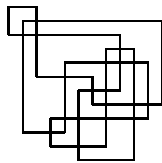


13n3013

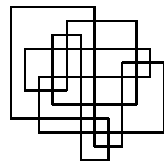
Almost alternating.



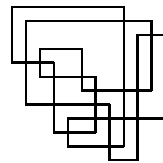
13n0613



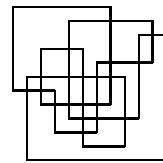
13n0635



13n0649



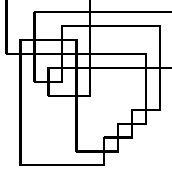
13n0714



13n4031

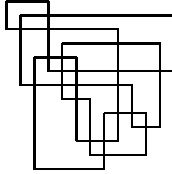
7.2. A partial list of 14 crossing knots with arc index 13. The 14 crossing knots in the lists below have Kauffman v -spread equal to 11, hence their arc index is at least 13. Using the methods described in the proofs of main theorems, we were able to find grid diagrams of them with 13 vertical arcs.

(2, 1)-nonalternating.

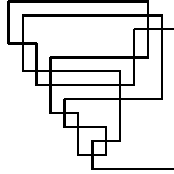


14n7534

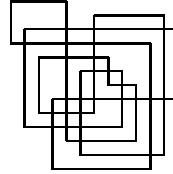
(3, 1)-nonalternating.



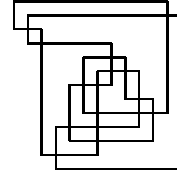
14n2637



14n10562

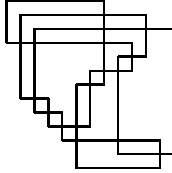


14n11853

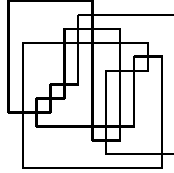


14n12211

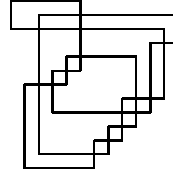
(4, 1)-nonalternating.



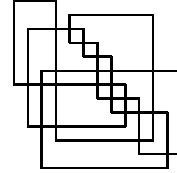
14n12930



14n24513

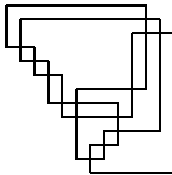


14n24551



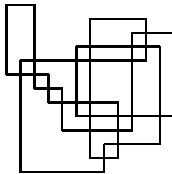
14n25035

(5, 1)-nonalternating.



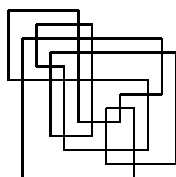
14n15965

2-nonalternating.



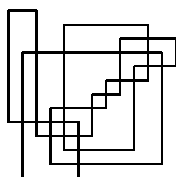
14n6923

3-nonalternating.



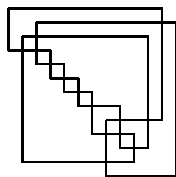
14n8036

4-nonalternating.



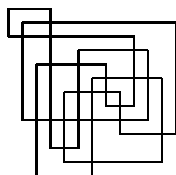
14n8863

6-nonalternating.



14n26177

Almost alternating.



14n21148

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